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Almost periodic solution of generalized Ginzburg-Landau equation

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Abstract One class of generalized Ginzburg-Landau equation is studied and the existence of almost periodic solution of the equation is proven when $f(t,x)$ is an almost periodic function of time t.

Keywords: **generalized Ginzburg-Landau equation, almost periodic solution, existence.**

We consider the Ginzburg-Landau equation as follows:

$$u_t = \alpha_0 u + \alpha_1 u_{xx} + \alpha_2 |u|^2 u + \alpha_3 |u|^2 u_x + \alpha_4 u^2 \bar{u}_x + \alpha_5 |u|^4 u + f,$$
$$t > 0,\ x \in \mathbb{R}^1, \tag{1}$$

where $\alpha_0 = a_0$ is a real number, $\alpha_j = a_j + \mathrm{i} b_j$, a_j and b_j are real numbers, $j = 1, 2, \cdots 5$. $u(t,x)$ is a complex value function of t and x. The equation is supplemented with the space-periodicity boundary condition and zero average condition as follows:

$$\begin{cases} u(t, x+L) = u(t,x),\ \forall t \in \mathbb{R}^1,\ \forall x \in \mathbb{R}^1, \\ \int_\Omega u \mathrm{d}x = 0, \end{cases} \tag{2}$$

where L is a positive number and $\Omega = (0, L) \subset \mathbb{R}^1$. Let $f(t,x)$ be an almost periodic function of t. Using the constructive method, we shall prove that Problem (1)-(2) has an almost periodic solution in time t. For the periodic solution of nonlinear evolution equations, see References [1,5,6]

In this paper, we have the following assumption:

Assumption 1. $a_1 > 0 > a_5$ and $-4a_1 a_5 > (b_3 - b_4)^2$.

1 Definitions and notations

Let X be a Banach space with the norm $\|\cdot\|_X$. For simplicity, we denote $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|$ as $p \neq 2$ and $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$. In addition, we denote "$\frac{\partial}{\partial t}$" by "$'$".

Definition 1 (**abstract almost periodic function**). Let $u(t)$ be a measurable function with value in a Banach space X. We say $u(t)$ is X-almost periodic if for any $\varepsilon > 0$ there exists a relatively dense set $E\{\varepsilon, u\} \subset \mathbb{R}\}^1$ such that

$$\operatorname*{ess\,sup}_{t\in\mathbb{R}^1} \| u(t+\tau) - u(t) \|_X \leqslant z, \qquad \forall \tau \in E\{e, u\}.$$

Let

$$L^p_{\mathrm{per}} = \{g \in L^p(\Omega), g(x) \text{ be an } L\text{-periodic function}\},$$

where the norm in L^p_{per} is defined just as in $L^p(\Omega)$;

$$H^k_{\mathrm{per}} = \{g \in H^k(\Omega),\ g(x) \text{ be an } L\text{-periodic function}\},$$

where the norm in H^k_{per} is defined just as in $H^k(\Omega)$;

$$L^p_{AP}(X) = \{g : \mathbb{R}^1 \to X, g(t) \in L^p(\mathbb{R}^1, X), \text{ and } g(t) \text{ is } X\text{-almost periodic}\}.$$

Let $<\cdot,\cdot>$ denote the duality relationship between H^{-1}_{per} and H^1_{per}. We define the linear operator $\mathscr{A}: H^1_{\mathrm{per}} \to H^{-1}_{\mathrm{per}}$ by the equation

$$\langle \mathscr{A}v, w\rangle = \int_\Omega (\alpha_1 \partial_x v \partial_x \bar{w} - \mu v \bar{w})\mathrm{d}x, \qquad \forall v, w \in H^1_{\mathrm{per}}.$$

Let $\rho(\mathscr{A})$ denote the regular set of $(\mathscr{A})$. We choose μ adequately such that $0 \in \rho\mathscr{A}$. Set

$$\mathscr{N}\phi = (\alpha_0 - \mu)\phi + \alpha_2 |\phi|^2\phi + \alpha_3 |\phi|^2\phi_x + \alpha_4 \phi^2\bar{\phi}_x + \alpha_5 |\phi|^4\phi, \ \forall \phi \in H^1_{\mathrm{per}}.$$

Then $\mathscr{N}$ is locally Lipschitz continuous, i.e. $\forall v, w \in H^1_{per}$, $\exists C > 0$, such that

$$\| \mathscr{N}(v) - \mathscr{N}(w) \| \leqslant C \| v - w \|_{H^1},$$

where the Lipschitz constant C depends continuously on $\| v \|_{H^1}$ and $\| w \|_{H^1}$.

Assumption 2. $f(t,x) \in L^\infty_{AP}(L^2_{\mathrm{per}})$, and $f'_t(t,x) \in L^\infty_{AP}(H^{-1}_{\mathrm{per}})$.

Let

$$M = \operatorname*{ess\,sup}_{t\in\mathbb{R}^1} \| f(t,x) \|, \qquad N = \operatorname*{ess\,sup}_{t\in\mathbb{R}^1} \| f'_t(t,x) \|_{H^{-1}}.$$

Using these notations, Problem (1)-(2) can be written formally in the form

$$u'(t) + \mathscr{A}u(t) = \mathscr{N}\hat{u}(t) + f(t,\cdot), \text{ for any } t,\ u(t) \in H^1_{r\mathrm{per}}. \tag{3}$$

Definition 2 (strong solution). $u(t,x)$ is said to be a strong solution of Problem (1)-(2) if the following conditions are satisfied:

(i) $u(t) \in L^2_{\mathrm{loc}}(\mathbb{R}^1, H^1_{\mathrm{per}})$,

(ii) $\mathscr{A}u(t) \in L^2_{\mathrm{loc}}(\mathbb{R}^1, L^2_{\mathrm{per}})$,

(iii) $u'(t) \in L^2_{\mathrm{loc}}(\mathbb{R}^1, L^2_{\mathrm{per}})$,

(iv) $u'(t,x) + \mathscr{A}u(t,x) = \mathscr{N}u(t,x) + f(t,x)$, a.e. on $\mathbb{R}^1 \times \Omega$.

The following inequalities will be used in the proofs later.

(i) Agmon's inequality:

$$\| u \|_{\infty} \leqslant K_1(\| u \|^2 + \| u_x \|^2)^{1/4} \| u \|^{1/2}, \tag{4}$$

where K_1 is a constant depending only on L.

(ii) Gagliardo-Nirenberg inequality for L^p and H^k spaces:

$$\| u \|_p \leqslant K(p,k,L) \| u \|_{H^1}^{\theta} \| u \|_q^{1-\theta}, \frac{1}{p} = \theta\left(\frac{1}{2} - k\right) + (1-\theta) \cdot \frac{1}{q}.$$

Specially, there exist positive constant numbers K_2 and K_3 depending only on L such that

$$\| u \|_6^3 \leqslant K_2^3(\| u \|^2 + \| \partial_x u \|^2)^{1/2} \| u \|^2 \tag{5}$$

and

$$\| u \|_{10}^5 \leqslant K_3^5(\| u \|^2 + \| \partial_x u \|^2) \| u \|^3. \tag{6}$$

(iii) Hölder's inequality:

$$\int_0^L | fg | \mathrm{d}x \leqslant \| f \|_p \| g \|_q$$

and Young's inequality

$$AB \leqslant A^p/p + B^q/q,$$

where A and B are non-negative real numbers, $1 < p < \infty$ and $1/p + 1/q = 1$. In Hölder's inequality, one may take $p = 1, q = \infty$.

(iv) For $k \geqslant 1$, we have

$$\| u \|_{\infty} \leqslant \sqrt{2} \| u \|^{1/2} \| \partial_x u \|^{1/2}, \qquad \forall\, u \in H_{\mathrm{per}}^k, \quad \int_{\Omega} u(x) \mathrm{d}x = 0.$$

Especially, there exists a positive constant number K_4 depending only on L such that

$$\| u \| \leqslant K_4 \| \partial_x u \|. \tag{7}$$

2 *A Priori* estimates

In this and the next section, we shall construct a bounded solution of Problem (1)-(2) by Galerkin's method and give *a priori* estimates for this solution.

Let $\{\phi_j\}$ be the basis of H_{per}^1 consisting of the eigenfunctions of $\mathscr{A}$, and consider the system of ordinary differential equations

$$(u_{m,r}'(t), \phi_j) + (\mathscr{A} u_{m,r}(t), \phi_j) = (\mathscr{N} \hat{u}_{m,r}(t), \phi_j) + (f, \phi_j),$$

$$j = 1,2,\cdots m \tag{8}$$

with

$$u_{m,r}(-r) = 0, \tag{9}$$

where $r \in \mathbb{R}^+$, $u_{m,r} = \sum_{j=1}^{m} \alpha_{m,r,j}(t)\phi_j$ and $\alpha_{m,r,j}(t)(j = 1,2,\cdots m)$ are undetermined functions. Since $(f(t), \phi_j)$ is continuous in t and $(\mathscr{N}\tilde{u}_{m,r}(t), \phi_j)$ is Lipschitz continuous in $(\alpha_{m,r,1}, \alpha_{m,r,2}, \cdots \alpha_{m,r,m})$, System (8)-(9) has a unique solution $(\alpha_{m,r,1}, \alpha_{m,r,2}, \cdots \alpha_{m,r,m})$, i.e. $u_{m,r}(t)$, on some interval $[-r, t_m]$. By the theory of ordinary differential equation, we can obtain global existence in $[-r, +\infty)$ when $\| u_{m,r} \|$ and $\| \partial_x u_{m,r} \|$ are uniformly bounded. Besides, *a priori* estimates in $\| u_{m,r} \|$, $\| \partial_x u_{m,r} \|$ and $\| u'_{m,r} \|$ can ensure that $\{u_{m,r}\}$ has a convergent subsequence and one can prove that the limit of the convergent subsequence of $\{u_{m,r}\}$ is the solution needed. Now, let us give *a priori* estimates in $\| u_{m,r} \|$ and $\| \partial_x u_{m,r} \|$.

Lemma 1. Under Assumptions 1 and 2, let a_1 be large enough to satisfy the inequality as follows:

$$a_1 > \frac{(b_3 - b_4)^2}{2 | a_5 |}.$$

Then there exists a positive constant C_1 depending only on $\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5$, L and f such that

$$\| u_{m,r} \| \leqslant C_1, \qquad \forall t \in [-r, +\infty).$$

Proof. Multiplying the jth equation of (8) by $\bar{\alpha}_{m,r,j}$, summing over j from 1 to m, and taking the real part on both sides of the result, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \| u_{m,r}(t) \|^2 \leqslant{}& a_0 \| u_{m,r}(t) \|^2 - a_1 \| \partial_x u_{m,r}(t) \|^2 + a_2 \int_\Omega | u_{m,r}(t) |^4 \mathrm{d}x \\ &+ a_5 \int_\Omega | u_{m,r}(t) |^6 \mathrm{d}x + \mathrm{Re}[((a_3 + a_4) - \mathrm{i}(b_3 - b_4)) \\ &\cdot \int_\Omega | u_{m,r}(t) |^2 u_{m,r}(t) \partial_x \bar{u}_{m,r}(t)] + \| f \| \cdot \| u_{m,r}(t) \|. \end{aligned}$$

By Ref. [2] and the assumption about a_1, we can choose constants $A_1 = \sqrt{|a_5|}$ and $A_2 = \sqrt{\frac{(b_3 - b_4)^2}{|a_5|}}$ which satisfy $A_1^2 A_2^2 = (b_3 - b_4)^2$ such that

$$\alpha = 2a_1 - A_2^2 > 0, \ \beta = -(2a_5 + A_1^2) > 0,$$

and the following inequality is true (see Inequality (29) in Ref. [2]):

$$\frac{\mathrm{d}}{\mathrm{d}t} \| u_{m,r}(t) \|^2 + \frac{1}{2} \| u_{m,r}(t) \|^2 \leqslant -\alpha \| \partial_x u_{m,r}(t) \|^2 - \frac{\beta}{6} \| u_{m,r}(t) \|_6^6$$

$$+ C'L + \frac{1}{2}\|f\|^2, \tag{10}$$

where $C' = 2^{7/2}|\alpha_0|^{3/2}/(3^{3/2}\beta^{1/2}) + 2^7|\alpha_2|^3/(27\beta^2) + 2/(3\beta^{1/2})$. Let $C_1 = 2C'L + M^2$. By Gronwall inequality, we have

$$\|u_{m,r}(t)\|^2 \leqslant C_1, \quad \forall t \in [-r, +\infty).$$

This completes the proof of Lemma 1.

Lemma 2. Under Assumptions 1 and 2, let a_1 be large enough to satisfy the inequality as follows:

$$a_1 > \max\left\{\sigma^*, \frac{(b_3 - b_4)^2}{2|\alpha_5|}\right\}, \tag{11}$$

where $\sigma^* = \frac{|\alpha_2|}{2}C_1K_2^3 + (|\alpha_3| + |\alpha_4|)K_1^2C_1^2 + |\alpha_5|C_1^4K_3^5 + 2C_1$. Then there exists a positive constant C_2 depending only on $\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, L$, and f such that

$$\|\partial_x u_{m,r}(t)\| \leqslant C_2, \ \forall t \in [-r, +\infty).$$

Proof. Let λ_j be the eigenvalue of the operator $\mathscr{A}$ and $\mathscr{A}\phi_j = \lambda_j\phi_j$, $j = 1,2,3,\cdots$. Multiplying the jth equation of (8) by $\bar{\lambda}_j$ and summing over j from 1 to m, then taking the real part on both sides of the result, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x u_{m,r}(t)\|^2 \leqslant & -a_1\|\partial_{xx}u_{m,r}(t)\|^2 + |\alpha_0|\,\|u_{m,r}(t)\|\cdot\|\partial_{xx}u_{m,r}(t)\| \\ & + |\alpha_2|\,\|u_{m,r}(t)\|_6^3\cdot\|\partial_{xx}u_{m,r}(t)\| + (|\alpha_3| \\ & + |\alpha_4|)\int_\Omega |u_{m,r}(t)||\partial_x u_{m,r}(t)||\partial_{xx}u_{m,r}(t)| \\ & + |\alpha_5|\,\|u_{m,r}(t)\|_{10}^5\cdot\|\partial_{xx}u_{m,r}(t)\| + \|f\|\cdot\|\partial_{xx}u_{m,r}(t)\|.\end{aligned} \tag{12}$$

By (4) ~ (6) and the inequality $\|u_x\|^2 \leqslant \|u\|\cdot\|u_{xx}\|$, we have the following inequalities:

$$-a_1\|\partial_{xx}u_{m,r}(t)\|^2 \leqslant -a_1\cdot\frac{1}{C_1}\|\partial_x u_{m,r}(t)\|^2\cdot\|\partial_{xx}u_{m,r}(t)\|. \tag{13}$$

$$\begin{aligned}|\alpha_2|\,\|u_{m,r}(t)\|_6^3\cdot\|\partial_{xx}u_{m,r}(t)\| & \leqslant |\alpha_2|K_2^3C_1^2(C_1^2 + \|\partial_x u_{m,r}(t)\|^2)^{1/2}\|\partial_{xx}u_{m,r}(t)\| \\ & \leqslant \frac{1}{2}|\alpha_2|K_2^3(C_1^2 + C_1^4)\|\partial_{xx}u_{m,r}(t)\| \\ & \quad + \frac{1}{2}|\alpha_2|K_2^3\|\partial_x u_{m,r}(t)\|^2\|\partial_{xx}u_{m,r}(t)\|.\end{aligned} \tag{14}$$

$$(|\alpha_3| + |\alpha_4|)\int_\Omega |u_{m,r}(t)|^2|\partial_x u_{m,r}(t)||\partial_{xx}u_{m,r}(t)|$$

$$\leqslant (|\alpha_3|+|\alpha_4|)\|u_{m,r}(t)\|_\infty^2 \cdot \|\partial_x u_{m,r}(t)\| \cdot \|\partial_{xx} u_{m,r}(t)\|$$

$$\leqslant (|\alpha_3|+|\alpha_4|)K_1^2 C_1 (C_1^2 + \|\partial_x u_{m,r}(t)\|^2)^{1/2} \cdot \|\partial_x u_{m,r}(t)\| \cdot \|\partial_{xx} u_{m,r}(t)\|$$

$$\leqslant (|\alpha_3|+|\alpha_4|)\frac{K_1^2 C_1^3}{2}\|\partial_{xx} u_{m,r}(t)\| \tag{15}$$

$$+ (|\alpha_3|+|\alpha_4|)K_1^2 C_1 \|\partial_x u_{m,r}(t)\|^2 \cdot \|\partial_{xx} u_{m,r}(t)\|,$$

$$|\alpha_5| \|u_{m,r}(t)\|_{10}^5 \cdot \|\partial_{xx} u_{m,r}(t)\| + \|f\| \cdot \|\partial_{xx} u_{m,r}(t)\|$$

$$\leqslant |\alpha_5| K_3^5 C_1^3 (C_1^2 + \|\partial_x u_{m,r}(t)\|^2)\|\partial_{xx} u_{m,r}(t)\|$$

$$\leqslant \alpha_5 | K_3^5 C_1^5 \|\partial_{xx} u_{m,r}(t)\| + |\alpha_5| K_3^5 C_1^3 \|\partial_x u_{m,r}(t)\|^2 \cdot \|\partial_{xx} u_{m,r}(t)\|. \tag{16}$$

Hence, Inequality (12) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x u_{m,r}(t)\|^2 \leqslant \Big\{\Big[M + |\alpha_0| C_1 + \frac{1}{2}|\alpha_2| K_2^3 (C_1^2 + C_1^4) + (|\alpha_3|+|\alpha_4|)\frac{K_1^2 C_1^3}{2} + |\alpha_5| K_3^5 C_1^5\Big] - \Big[a_1 \cdot \frac{1}{C_1} - \frac{1}{2}|\alpha_2| K_2^3 - (|\alpha_3| + |\alpha_4|)K_1^2 C_1 - |\alpha_5| K_3^5 C_1^3\Big]\|\partial_x u_{m,r}(t)\|^2\Big\} \cdot \|\partial_{xx} u_{m,r}(t)\|.$$

Taking

$$C_2 = \sqrt{M + |\alpha_0| C_1 + \frac{1}{2}|\alpha_2| K_2^3 (C_1^2 + C_1^4) + (|\alpha_3|+|\alpha_4|)K_1^2 C_1^3 + |\alpha_5| {K_3}^5 {C_1}^5},$$

we have

$$\|\partial_x u_{m,r}(t)\| \leqslant C_2, \forall t \in [-r, +\infty).$$

This completes the proof of Lemma 2.

Remark 1. Since C_1 is independent of a_1, a_1 can be taken to satisfy Inequality (10). Set

$$\sigma_1 = \max\left\{\sigma^*, \frac{(b_3 - b_4)^2}{2|\alpha_5|}\right\}.$$

Lemma 3. Under Assumptions 1 and 2, let a_1 be large enough such that $a_1 > \sigma_1$. Then there exists a positive const ant C_3 depending only on α_0, α_1, α_2, α_3, α_4, α_5, L, and f such that

$$\int_t^{t+1}\|\partial_{xx} u_{m,r}(t)\| \leqslant C_3, \qquad \forall t \in [-r, +\infty).$$

Proof. Let λ_j be the eigenvalue of the operator $\mathscr{A}$ and $\mathscr{A}\phi_j = \lambda_j \phi_j$, $j = 1, 2, 3, \cdots$. Multiplying the jth equation of (8) by $\bar{\lambda}_j$, summing over j from 1 to m, and taking the real part on both sides of the result, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x u_{m,r}(t)\|^2 + a_1\|\partial_{xx}^2 u_{m,r}(t)\|^2$$

$$= a_0\|\partial_x u_{m,r}(t)\|^2 + \mathrm{Re}\Big(\alpha_2\int_\Omega\Big[u_{m,r}^2(t)\big(\partial_x\bar{u}_{m,r}(t)\big)^2 + 2|u_{m,r}|^2(\partial_x u_{m,r})^2\Big]\Big)$$

$$-\mathrm{Re}\Big(\alpha_3\int_\Omega |u_{m,r}(t)|^2\partial_x u_{m,r}(t)\partial_{xx}\bar{u}_{m,r}(t)\Big)$$

$$+\mathrm{Re}\Big(\alpha_4\int_\Omega |\partial_x u_{m,r}(t)|^2 u_{m,r}(t)\partial_x\bar{u}_{m,r}(t)\Big) + 3a_5\int_\Omega |u_{m,r}(t)|^4|\partial_x u_{m,r}(t)|^2$$

$$+2\mathrm{Re}\Big(\alpha_5\int_\Omega |u_{m,r}^2(t)|u_{m,r}^2(t)\partial_x\bar{u}_{m,r}^2(t)\Big) + \mathrm{Re}(f,\partial_{xx}u_{m,r}(t)).$$

By Agmom's inequality, Gagliarde-Nirenberg inequality[3], and Lemmas 1 and 2, we have the following inequalities:

$$|\mathrm{Re}(f,\partial_{xx}u_{m,r}(t))|$$

$$\leqslant \varepsilon\|\partial_{xx}u_{m,r}(t)\|^2 + \frac{1}{4\varepsilon}\|f\|^2, \Big|\mathrm{Re}\Big(\alpha_2\int_\Omega[u_{m,r}^2(t)(\partial_x\bar{u}_{m,r}(t))^2 + 2|u_{m,r}|^2(\partial_x u_{m,r})^2]\Big)\Big|$$

$$\leqslant 3|\alpha_2|\,\|u_{m,r}(t)\|_\infty^2\|\partial_x u_{m,r}(t)\|^2 \leqslant 3|\alpha_2|K_1^2C_1C_2^2(C_1^2 + C_2^2)^{\frac{1}{2}},$$

$$\Big|-\mathrm{Re}\Big(\alpha_3\int_\Omega |u_{m,r}(t)|^2\partial_x u_{m,r}(t)\partial_{xx}\bar{u}_{m,r}(t)\Big)\Big|$$

$$\leqslant |\alpha_3|\,\|u_{m,r}(t)\|_\infty^2\|\partial_x u_{m,r}(t)\|\cdot\|\partial_{xx}u_{m,r}(t)\|$$

$$\leqslant \varepsilon\|\partial_{xx}u_{m,r}(t)\|^2 + \frac{1}{4\varepsilon}|\alpha_3|^2K_1^4C_1^2C_2^2(C_1 + C_2)^2,$$

$$\Big|\mathrm{Re}\Big(\alpha_4\int_\Omega|\partial_x u_{m,r}(t)|^2u_{m,r}(t)\partial_x\bar{u}_{m,r}(t)\Big)\Big|$$

$$\leqslant |\alpha_4|\,\|\partial_x u_{m,r}(t)\|_6^3\cdot\|u_{m,r}(t)\| \leqslant |\alpha_4|K_2^3C_1C_2^2(C_2^2 + \|\partial_{xx}u_{m,r}(t)\|^2)^{1/2}$$

$$\leqslant \varepsilon\|\partial_{xx}u_{m,r}(t)\|^2 + \frac{1}{4\varepsilon}|\alpha_4|^2K_2^6C_1^2C_2^4 + \varepsilon C_2^2,$$

$$\Big|3a_5\int_\Omega|u_{m,r}(t)|^4|\partial_x u_{m,r}(t)|^2\Big|$$

$$\leqslant 3|a_5|\,\|u_{m,r}(t)\|_\infty^4\cdot\|\partial_x u_{m,r}(t)\|^2 \leqslant 3|a_5|K_1^4C_1^2C_2^2(C_1^2 + C_2^2).$$

Similarly,

$$\left|2\mathrm{Re}\left(\alpha_5\int_\Omega |u_{m,r}(t)|^2 u_{m,r}^2(t)\partial_x \bar{u}_{m,r}^2(t)\right)\right| \leqslant 2|\alpha_5| K_1^4 C_1^2 C_2^2 (C_1^2+C_2^2).$$

Choosing ε small enough, there exists a positive constant D^* such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x u_{m,r}(t)\|^2 + \frac{a_1}{2}\|\partial_{xx} u_{m,r}(t)\|^2 \leqslant D^*,$$

where D^* depends only on α_0, α_1, α_2, α_3, α_4, α_5, L, and f. Letting

$$C_3 = \frac{2D^* + C_2^2}{a_1},$$

we have

$$\int_t^{t+1}\|\partial_{xx} u_{m,r}(t)\|^2 \leqslant C_3, \ \forall\, t \in [-r, +\infty).$$

This completes the proof of Lemma 3.

Lemma 4. Under Assumptions 1 and 2, let a_1 be large enough such that

$$a_1 > \max\{\sigma_1, \sigma_2\}, \tag{17}$$

where

$$\sigma_2 = (|\alpha_3|+|\alpha_4|)K_1^2K_4^2\left(1+\frac{1}{K_4^2}\right)(C_1^2+C_2^2)$$
$$+\ a_0K_4^2 + 6|\alpha_2|K_4^2C_1C_2 + 20|\alpha_5|K_4^2C_1^2C_2^2 + 2K_4^2.$$

Then there exists a positive constant C_4 depending only on α_0, α_1, α_2, α_3, α_4, α_5, L, and f such that

$$\|u'_{m,r}(t)\| \leqslant C_4, \ \forall\, t \in [-r, +\infty).$$

Proof. From (8), we have

$$(u''_{m,r}(t), \phi_j) + (\mathscr{A}u'_{m,r}(t), \phi_j) = ([\mathscr{N}(u_{m,r}(t))]', \phi_j) + \langle f'_t, \phi_j\rangle,\ j = 1,2,\cdots m. \tag{18}$$

Multiplying the jth equation of (18) by $\bar{\alpha}'_{m,r,j}(t)$, summing over j from 1 to m, and taking the real part on both sides of the result, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u'_{m,r}(t)\|^2 \leqslant -a_1\|\partial_x u'_{m,r}(t)\|^2 + a_0\|u'_{m,r}(t)\|^2 + 3|\alpha_2|\,\|u_{m,r}(t)\|_\infty^2$$
$$\cdot\ \|u'_{m,r}(t)\|^2 + (|\alpha_3|+|\alpha_4|)\cdot\left[2\int_\Omega |u'_{m,r}(t)|^2|u_{m,r}(t)|\right.$$
$$\left.|\partial_x u_{m,r}(t)| + \int_\Omega |u_{m,r}(t)|^2|\partial_x u'_{m,r}(t)||u'_{m,r}(t)|\right]$$

$$+ 5 | \alpha_5 | \, \| u_{m,r}(t) \|_\infty^4 \cdot \| u'_{m,r}(t) \|^2 + \| f \|_{H^{-1}} \cdot \| u'_{m,r}(t) \| .$$

Noting

$$2\int_\Omega | u'_{m,r}(t) |^2 | u_{m,r}(t) | | \partial_x u_{m,r}(t) |$$

$$\leqslant \| u'_{m,r}(t) \|_\infty^2 \cdot \| u_{m,r}(t) \| \cdot \| \partial_x u_{m,r}(t) \|$$

$$\leqslant K_1^2 C_1 C_2 \| u'_{m,r}(t) \|^2 + \frac{K_1^2}{2} C_1 C_2 \| \partial_x u'_{m,r}(t) \|^2$$

and

$$\int_\Omega | u_{m,r}(t) |^2 | \partial_x u'_{m,r}(t) | | u'_{m,r}(t) |$$

$$\leqslant \| u_{m,r}(t)_\infty^2 \cdot \| \partial_x u'_{m,r}(t) \| \cdot \| u'_{m,r}(t) \|$$

$$\leqslant \frac{K_1^2}{2} C_1 \sqrt{C_1^2 + C_2^2} \, \| u'_{m,r}(t) \|^2 + \frac{K_1^2}{2} C_1 \sqrt{C_1^2 + C_2^2} \| \partial_x u'_{m,r}(t) \|^2,$$

and using Inequality (7), we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| u'_{m,r}(t) \|^2$$

$$\leqslant - \frac{a_1}{K_4^2} \| u'_{m,r}(t) \|^2 + a_0 \| u'_{m,r}(t) \|^2 + 3 | \alpha_2 | \cdot 2 C_1 C_2 \cdot \| u'_{m,r}(t) \|^2$$

$$+ (| \alpha_3 | + | \alpha_4 |) \Big[\Big(K_1^2 C_1 C_2 + \frac{K_1^2}{2} C_1 \sqrt{C_1^2 + C_2^2} \Big) \| u'_{m,r}(t) \|^2$$

$$+ \Big(\frac{K_1^2}{2} C_1 C_2 + \frac{K_1^2}{2} C_1 \sqrt{C_1^2 + C_2^2} \Big) \| \partial_x u'_{m,r}(t) \|^2 \Big]$$

$$+ 5 | \alpha_5 | \cdot 4 C_1^2 C_2^2 \cdot \| u'_{m,r}(t) \|^2 + N \cdot \| u'_{m,r}(t) \|$$

$$\leqslant \Big\{ N - \Big[\frac{a_1}{K_4^2} - (| \alpha_3 | + | \alpha_4 |) \, K_1^2 \Big(1 + \frac{1}{K_4^2} \Big) (C_1 C_2 + \frac{1}{2} C_1 \sqrt{C_1^2 + C_2^2})$$

$$- a_0 - 6 | \alpha_2 | \, C_1 C_2 - 20 | \alpha_5 | \, C_1^2 C_2^2 \Big] \| u'_{m,r}(t) \| \cdot \| u'_{m,r}(t) \| \Big\}.$$

Setting

$$C_4 = \frac{N}{2},$$

we have

$$\| u'_{m,r}(t) \| \leqslant C_4, \ \forall t \in [-r, +\infty).$$

This completes the proof of Lemma 4.

Remark 2. Since C_1 and C_2 are independent of a_1, a_1 can be taken to satisfy Inequality (17).

3 Bounded solution

Theorem 1. Under Assumptions 1 and 2, let a_1 be large enough to satisfy the inequality as follows:

$$a_1 > \max\{\sigma_1, \sigma_2\}.$$

Then Problem (1)-(2) has a strong solution u which satisfies these inequalities as follows:

$$\| u(t) \| \leqslant C_1, \ \| \partial_x u(t) \| \leqslant C_2, \ \int_t^{t+1} \| \partial_{xx} u(t) \|^2 \leqslant C_3, \ \| u'(t) \| \leqslant C_4, \tag{19}$$

where C_1, C_2, C_3 and C_4 are given in Lemmas 1 ~ 4, respectively.

Proof. Using Lemmas 1 ~ 4, by the standard compactness arguments, we can take an appropriate subsequence which we denote also by $\{u_{m,r}\}$, such that

$$\lim_{r\to+\infty} u_{m,r}(t) = u_m(t) \quad \text{weakly star in } L^\infty(\mathbb{R}^1, H^1_{\text{per}}),$$

$$\lim_{r\to+\infty} u_{m,r}(t) = u_m(t) \quad \text{strongly in } L^\infty_{\text{loc}}(\mathbb{R}^1, L^2_{\text{per}}),$$

$$\lim_{r\to+\infty} u'_{m,r}(t) = u'_m(t) \quad \text{weakly star in } L^\infty(\mathbb{R}^1, L^2_{\text{per}}),$$

$$\lim_{r\to+\infty} \mathscr{A} u_{m,r}(t) = \mathscr{A} u_m(t) \quad \text{weakly in } L^\infty_{\text{loc}}(\mathbb{R}^1, L^2_{\text{per}}),$$

$$\lim_{r\to+\infty} u_{m,r}(t,x) = u_m(t,x) \quad \text{a.e. in } \Omega \times \mathbb{R}^1,$$

$$\lim_{r\to+\infty} | u_{m,r}(t) |^2 u_{m,r}(t) = | u_m(t) |^2 u_m(t) \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^1, L^2_{\text{per}}),$$

and

$$\lim_{r\to+\infty} | u_{m,r}(t) |^4 u_{m,r}(t) = | u_m(t) |^4 u_m(t) \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^1, L^2_{\text{per}}).$$

Thus $u_m(t)$ is the solution of the following equation:

$$(u'_m(t), \phi_j) + (\mathscr{A} u_m(t), \phi_j) = (\mathscr{N} \hat{u}_m(t), \phi_j) + (f(t), \phi_j),$$

$$j = 1, 2, \cdots m, \tag{20}$$

where $u_m(t) = \sum_{j=1}^m \alpha_{m,j}(t) \phi_j$. Besides, those inequalities in (19) are also valid for $u = u_m$.

Similarly, we can again take a subsequence of $\{u_m\}$, which is denoted also by $\{u_m\}$, such that

$$\lim_{m\to+\infty} u_m(t) = u(t) \quad \text{weakly star in } L^\infty(\mathbb{R}^1, H^1_{\text{per}}),$$

$$\lim_{m\to+\infty} u_m(t) = u(t) \qquad \text{strongly in } L^\infty_{loc}(\mathbb{R}^1, L^2_{per}),$$

$$\lim_{m\to+\infty} u'_m(t) = u'(t) \qquad \text{weakly star in } L^\infty(\mathbb{R}^1, L^2_{per}),$$

$$\lim_{m\to+\infty} \mathscr{A}u_m(t) = \mathscr{A}u(t) \qquad \text{weakly in } L^2_{loc}(\mathbb{R}^1, L^2_{per}),$$

$$\lim_{m\to+\infty} u_m(t,x) = u(t,x) \qquad \text{a.e. in } \Omega\times\mathbb{R}^1,$$

$$\lim_{m\to+\infty} |u_m(t)|^2 u_m(t) = |u(t)|^2 u(t) \qquad \text{weakly in } L^2_{loc}(\mathbb{R}^1, L^2_{per}),$$

and

$$\lim_{m\to+\infty} |u_m(t)|^4 u_m(t) = |u(t)|^4 u(t) \qquad \text{weakly in } L^2_{loc}(\mathbb{R}^1, L^2_{per}).$$

Besides, $u(t)$ satisfies (20) for $j = 1,2,3,\cdots$, or

$$u'(t,x) + \mathscr{A}u(t,x) = \mathscr{N}\hat{u}(t,x) + f(x,t), \quad \text{a.e. on } \mathbb{R}^1\times\Omega,$$

and those inequalities in (19) are true. Thus $u(t)$ is a required strong solution of Problem (1)-(2). This completes the proof of Theorem 1.

4 Existence of almost periodic solution

In Sec. 3, we obtain a bounded solution $u(t)$ of Problem (1)-(2). We shall prove that this solution has almost periodicity. For this, we need to prepare an inequality about the nonlinear operator $\mathscr{N}$.

By (20), we have

$$\begin{cases}(u'_m(t), \phi_j) + (\mathscr{A}u_m(t),\phi_j) = (\mathscr{N}\hat{u}_m(t),\phi_j) + (f(t),\phi_j)\\ (u'_m(t+\tau), \phi_j) + (\mathscr{A}u_m(t+\tau),\phi_j) = (\mathscr{N}\hat{u}_m(t+\tau),\phi_j) + (f(t+\tau),\phi_j)\end{cases} \tag{21}$$

for $j = 1,2,3,\cdots$ where $\tau\in\mathbb{R}^1$. In the proof below, we need to consider the boundedness of the following term:

$$T = \| [\mathscr{N}\hat{u}_m(t+\tau) - (a_0-\mu)u_m(t+\tau)] - [\mathscr{N}\hat{u}_m(t) - (a_0-\mu)u_m(t)] \|.$$

Since

$$\begin{aligned}&\| |u_m(t+\tau)|^2 u_m(t+\tau) - |u_m(t)|^2 u_m(t) \|\\ &\leqslant (\|u_m(t+\tau)\|^2_\infty + \|u_m(t+\tau)\|_\infty \cdot \|u_m(t)\|_\infty\\ &\quad + \|u_m(t)\|^2_\infty)\cdot \|u_m(t+\tau) - u_m(t)\|\\ &\leqslant 6C_1C_2K_4 \|\partial_x(u_m(t+\tau) - u_m(t))\|,\end{aligned} \tag{22}$$

$$\| |u_m(t+\tau)|^2\partial_x u_m(t+\tau) - |u_m(t)|^2\partial_x u_m(t)\|$$

$$\leqslant \|u_m(t+\tau)\|^2_\infty \cdot \|\partial_x(u_m(t+\tau) - u_m(t))\|$$

$$+ (\| u_m(t+\tau) \|_\infty + \| u_m(t) \|_\infty) \| \partial_x u_m(t) \| \cdot \| u_m(t+\tau) - u_m(t) \|_\infty$$

$$\leqslant (C_1 + \sqrt{2} C_1 C_2 + \sqrt{2} C_1 C_2 K_4) \| \partial_x (u_m(t+\tau) - u_m(t)) \| . \tag{23}$$

Similarly,

$$\| u_m^2(t+\tau) \partial_x \bar{u}_m(t+\tau) - u_m^2(t) \partial_x \bar{u}_m(t) \|$$

$$\leqslant (C_1 + \sqrt{2} C_1 C_2 + \sqrt{2} C_1 C_2 K_4) \| \partial_x (u_m(t+\tau) - u_m(t)) \| , \tag{24}$$

$$\| \, | u_m(t+\tau) |^4 u_m(t+\tau) - | u_m(t) |^4 u_m(t) \|$$

$$\leqslant (\| u_m(t+\tau) \|_\infty^4 + \| u_m(t+\tau) \|_\infty^3 \cdot \| u_m(t) \|_\infty + \| u_m(t+\tau) \|_\infty \cdot \| u_m(t) \|_\infty^3$$

$$+ \| u_m(t+\tau) \|_\infty^2 \cdot \| u_m(t) \|_\infty^2 + \| u_m(t) \|_\infty^4) \cdot \| u_m(t+\tau) - u_m(t) \|$$

$$\leqslant 20 C_1^2 C_2^2 K_4 \| \partial_x (u_m(t+\tau) - u_m(t)) \| . \tag{25}$$

For convenience, set

$$\sigma^{**} = 6 C_1 C_2 K_4 + 2(C_1 + \sqrt{2} C_1 C_2 + \sqrt{2} C_1 C_2 K_4) + 20 C_1^2 C_2^2 .$$

Then

$$T \leqslant \sigma^{**} \| \partial_x (u_m(t+\tau) - u_m(t)) \| .$$

Now we give the main theorem in this paper as follows.

Theorem 2. Under Assumptions 1 and 2, let a_1 be large enough to satisfy the inequality as follows:

$$a_1 > \max\{ \sigma_1, \sigma_2, \sigma_3 \},$$

where $\sigma_3 = (a_0 + \sigma^{**}) K_4$. Then Problem (1)-(2) has an L^2-almost periodic solution.

Proof. Let $u(t)$ be the bounded solution of Problem (1)-(2) given by Theorem 1. One needs to prove that the solution is L^2-almost periodic. In fact, since $\{ u_m(t) \}$ converges uniformly to $u(t)$ on $\mathbb{R}^1$, it is sufficient to prove that $u_m(t)$ has almost periodicity for each $m \in \{1,2,3,\cdots\}$. By Assumption 2, f is almost periodic, then for any $\varepsilon > 0$ there is a relatively dense set $E\{\varepsilon, f\}$ such that

$$\| f(t+\tau) - f(t) \| \leqslant \varepsilon, \text{ for } \tau \in E\{\varepsilon, f\}. \tag{26}$$

From (18), for any $\tau \in E\{\varepsilon, f\}$, we have

$$((u_m'(t+\tau) - u_m'(t)), \phi_j) + (\mathscr{A}(u_m(t+\tau) - u_m(t)), \phi_j)$$

$$= (\mathscr{N} u_m(t+\tau) - \mathscr{N} u_m(t), \phi_j) + (f(t+\tau) - f(t), \phi_j),$$

$$j = 1,2,\cdots,m. \tag{27}$$

Multiplying the jth equation of (20) by $\bar{\alpha}_{m,j}(t+\tau) - \bar{\alpha}_{m,j}(t)$ and summing over j from 1 to m, then taking the real part on both sides of the result yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\| u_m(t+\tau) - u_m(t) \|^2 \\
\leqslant & - a_1 \| \partial_x [u_m(t+\tau) - u_m(t)] \|^2 + a_0 \| u_m(t+\tau) - u_m(t) \|^2 \\
& + T \| u_m(t+\tau) - u_m(t) \| + \| f(t+\tau) - f(t) \| \cdot \| u_m(t+\tau) - u_m(t) \| \\
\leqslant & - \frac{a_1}{K_4} \| u_m(t+\tau) - u_m(t) \| \cdot \| \partial_x [u_m(t+\tau) - u_m(t)] \| \\
& + a_0 \| u_m(t+\tau) - u_m(t) \| \cdot \| \partial_x [u_m(t+\tau) - u_m(t)] \| \\
& + \sigma^{**} \| u_m(t+\tau) - u_m(t) \| \cdot \| \partial_x [u_m(t+\tau) - u_m(t)] \| \\
& + \varepsilon \cdot K_4 \| \partial_x [u_m(t+\tau) - u_m(t)] \| \\
= & \left\{ \varepsilon \cdot K_4 - \left[\frac{a_1}{K_4} - (a_0 + \sigma^{**}) \right] \| u_m(t+\tau) - u_m(t) \| \right\} \cdot \| \partial_x [u_m(t+\tau) - u_m(t)] \|
\end{aligned}$$

From this, we obtain

$$\| u_m(t+\tau) - u_m(t) \| \leqslant \frac{K_4}{\frac{a_1}{K_4} - (a_0 + \sigma^{**})} \cdot \varepsilon.$$

Thus $u_m(t)$ is L^2-almost periodic[4]. This completes the proof of Theorem 2.

Remark 3. Since σ_3 is independent of a_1, a_1 can be taken to satisfy the condition in Theorem 2.

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